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Transfer Function Models of Inventory Policies and Bullwhip Quantification in Supply Chain

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Abstract

Control theory approach to the supply chain is an analytical method to quantify the bullwhip effect. In this study, using control theory approach the bullwhip effect inducing nature of various inventory policies is analyzed. The inventory policies considered are forecast based, forecast + order smoothing, Order-Up-To (OUT), OUT + net stock smoothing + on-order inventory smoothing, and OUT + net stock smoothing + on-order inventory smoothing + order smoothing. The performance measures considered are order rate variance ratio and bullwhip slope. The forecast based policies cause the de-whip effect in the supply chain. The OUT policy and its variants cause bullwhip generation in the supply chain. It is found that the order variance amplification is more under OUT policy than its variants.

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Keywords: Control theory; Transfer function; Supply chain; Inventory policy; Bullwhip effect

1. Introduction

In a supply chain, the increase in *order* variability as one moves from a downstream stage to an upstream stage is called bullwhip effect [1]. In general, the bullwhip quantification methods can be classified into three categories: experimentation [2], simulation analysis [3] and analytical method [4]. The control system approach is an analytical method to quantify the bullwhip effect in supply chains [5,6,7,8]. As per control system engineering concept, "transfer functions are used to connect the input-output relations in systems that can be described by a linear, time-

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Nomenclature

α	Exponential smoothing parameter	L	Delivery lead time
β	Net stock smoothing parameter	T	Review period
γ	On-order inventory smoothing parameter	k	Order lead time
δ	Order smoothing parameter	L_T	Total lead time
h	Desired service level	IP_t	Inventory position at period t
t	Period index	S_t	Order-up-to level at period t
D_t	Customer demand at period t	OOI_t	On-order inventory at period t
F_t	Forecasted demand at period t	NS_t	Net stock at period t
$\hat{\sigma}_t$	Estimated standard deviation	A	Amplitude
O_t	Order placed at period t	U	Amplitude ratio
p	Safety factor	ω	Frequency
i	Supply chain stage index	z	z-transform operator
m^S	Number of stages	E^i	Position of stage i
Var^i	Variance of orders placed by stage i	OVR	Order rate variance ratio
Var^C	Variance of orders placed by customer	$BwSl$	Bullwhip slope
N	Number of periods		

invariant and differential equations" [9]. The transfer function can be represented as the ratio of the Laplace transform of the output from the system to the Laplace transform of the input to the system. Simon [10] initiated the application of the transfer function approach to inventory control management and is mainly concerned with the development of a methodology. Later, the application of the control system approach for production and inventory control system with z-transform model, a special case of Laplace transform, is proposed by various researchers [11,12,13,14]. Disney and Towill [15] pointed out the specific reasons for the supremacy of transfer function approach for system analysis and these reasons are as follows: use of standard forms, block diagram representation, frequency domain calculations, transformation from one domain to another (specifically conversion from time domain analysis to frequency domain analysis helps better understanding of the system properties), use of single transfer function to model relationship of two systems, immediate identification of system structure, application to stochastic system, and integration with simulation approach.

The supply chain can be considered as an input-output system with supply chain members as system components, and they are connected as per the structure of the supply chain [5]. The customer demand denotes the input to the members in the first stage, and the order quantity represents output from the members, which is again the input to the members in the second stage. The input-output view of supply chain helps to make an analogy with control system engineering theory. Dejonckheere et al. [5,6,7] investigated the bullwhip effect created by Order-Up-To (OUT) policy using control system engineering theory. Block diagram representation of OUT and variant of OUT policies under different forecasting methods and its reduction to derive the transfer function for order quantity determination were studied in detail. Specifically, Dejonckheere et al. [6] measured variance amplification order in a simple two-stage serial supply chain where the retailers are facing a real demand pattern. The methodology adopted is the transfer function analysis of inventory policy followed (OUT policy) by the members of the supply chain. Bullwhip effect is quantified, and it is observed that the OUT policy and its smoothed variants cause order variance amplification in supply chains under different forecasting methods such as exponential smoothing and moving average. Dejonckheere et al. [7] quantified bullwhip effect in a four-stage serial supply chain (retailer, wholesaler, distributor, and factory) using control system engineering concepts. The transfer functions are derived for OUT and variants of OUT with smoothing rule under exponential and moving average forecasting methods. They also proposed the transfer function for any stage of the supply chain from the single level transfer function. Jakšič and Rusjan [8] studied the effect of various inventory policies in creating the bullwhip effect using transfer function approach. They quantified the increase in standard deviation of orders placed in each stage of a simple serial two-stage supply chain under different inventory policies.

2. Control theory approach

Control theoretic approach to supply chains involves three steps: (i) Derivation of transfer functions for inventory policies (ii) Preparation of frequency response plots and (iii) Spectral analysis of demand patterns [6]. The first two steps support the 'identification' of bullwhip effect creation nature of inventory policy followed by the members of the supply chain. The inclusion of the third step is to 'quantify' the bullwhip generated in the supply chain.

2.1. Derivation of transfer functions for inventory policies

The transfer function is termed as the ratio of z-transform of the output from the system and to the z-transform of the input to the system, where z-transform is a special case of the Laplace transform. The input to the system is the customer demand as per the demand pattern, and output from the system is the order quantity determined based on the inventory policy. For a supply chain, the transfer functions can be derived for the inventory policies followed by the members of the supply chain. An inventory policy refers to the rule to be used to make decisions on when and how much to order. One of the reasons of bullwhip effect generation in a supply chain is the inventory policy used [12]. In general, the inventory control system can be classified as periodic review system and continuous review system. In the periodic review system, the inventory position is reviewed at regular intervals (at review periods), and an appropriate quantity is ordered. However, in the continuous review system, reviews are often carried out continuously, and a fixed quantity of the item is ordered when the inventory position reaches the reorder level. The most widely used periodic review type inventory policy is the OUT policy and in a supply chain, the magnitude of bullwhip effect is large under this policy [6]. We have considered five inventory policies such as forecast based policy, order smoothing policy, OUT policy and two variants of OUT policy. The above policies are adapted from the study of Jakšič and Rusjan [8] and they derived these policies from the general linear inventory policy proposed by Bowman [16]. All policies are periodic review type and have a constant review interval of unit period (i.e. T = 1). Simple exponential smoothing is used to forecast demand for the next period, and the equation is as follows:

$$F_t = F_{t-1} + \alpha(D_t - F_{t-1}) \tag{1}$$

- Policy-1 (forecast based): The order quantity for the period t is equal to the demand forecasted for the period t.

$$O_t = F_t \tag{2}$$

- Policy-2 (forecast + order quantity smoothing): This policy is a modification of previous policy. An order smoothing parameter (δ) is introduced to smooth the difference of order placed in period t-1 and demand forecasted for the period t.

$$O_t = F_t + (1-\delta)(O_{t-1} - F_t) \tag{3}$$

- Policy-3 (OUT): In this policy, an order is placed at each review period so that the sum of the inventory position and the order placed in that period must be equal to the order-up-to level. The order-up-to level is the sum of the demand for risk period and safety stock level. The risk period demand is the total demand for the lead time (order lead time (k) + delivery lead time (L)) and the review period (T). Safety stock is an estimate of the standard deviation of demand forecasted for the risk period ($T + k + L$). So the order-up-to level can be written as:

$$S_t = F_t\,(T + (k + L)) + h\hat{\sigma}_t \sqrt{T + (k + L)} \tag{4}$$

By substituting $T = 1$ and $h\hat{\sigma}_t = pF_t$ in equation 4,

$$S_t = F_t\,(1 + (k + L)) + pF_t \sqrt{1 + (k + L)} \tag{5}$$

Inventory position at a time point shows the net of on-hand, on-order and backorder inventories. That is, Inventory position = (On-hand inventory) + (On-order inventory) – Backorders. On-hand inventory is the quantity immediately available to meet the demand. On-order inventory is the quantity ordered but not yet received and backorders show the demand that is not yet met. So the inventory position calculation is as follows:

$$IP_t = NS_t + OOI_t \tag{6}$$

The order quantity is determined as:

$$O_t = S_t - IP_t \tag{7}$$

By substituting equations 5 and 6 in equation 7,

$$O_t = F_t\,(1+(k+L)) + pF_t\,\sqrt{1+(k+L)} - \left[NS_t + OOI_t\right] \tag{8}$$

- Policy-4 (OUT + net stock smoothing + on-order inventory smoothing): This policy is a variant of OUT policy. Net stock smoothing parameter (β) and on-order inventory smoothing parameter (γ) are introduced to smooth order quantity.

$$O_t = F_t + \beta\left[pF_t\,\sqrt{1+(k+L)} - NS_t\right] + \gamma\left[F_t\,(k+L) - OOI_t\right] \tag{9}$$

- Policy-5 (OUT + net stock smoothing + on-order inventory smoothing + order quantity smoothing): This policy is also another variant of OUT policy and which is formulated by introducing order smoothing parameter to the Policy-4.

$$O_t = F_t + \beta\left[pF_t\,\sqrt{1+(k+L)} - NS_t\right] + \gamma\left[F_t\,(k+L) - OOI_t\right] - (1-\delta)\left(O_{t-1} - F_t\right) \tag{10}$$

Initially, we have formulated a control engineering block diagram representation of each of the inventory policy. Corresponding to each block diagram, signal flow graph are developed and then well-known Mason's gain formula is applied to derive the transfer function of each inventory policy and which is shown in Table 1.

Table 1. Transfer functions of inventory policies

Sl. No.	Inventory policy	Transfer function
1	Policy-1 (Forecast based)	$\dfrac{\alpha z}{z-(1-\alpha)}$
2	Policy-2 (Forecast + order quantity smoothing)	$\dfrac{\alpha\delta z^2}{[z-(1-\alpha)][z-(1-\delta)]}$
3	Policy-3 (OUT)	$\dfrac{z\left(1+\alpha+\alpha(k+L)+\alpha p\sqrt{1+k+L}\right)-\left(1+\alpha(k+L)+\alpha p\sqrt{1+k+L}\right)}{z-(1-\alpha)}$
4	Policy-4 (OUT + net stock smoothing + on-order inventory smoothing)	$\dfrac{z^{L+1}\left[\beta[z-(1-\alpha)]+\alpha(z-1)[1+\beta p\sqrt{1+k+L}+\gamma(k+L)]\right]}{z^2 z^L - z\left[z^L(2-\alpha-\gamma)+\gamma-\beta\right]+(1-\alpha)\left[z^L(1-\gamma)+\gamma-\beta\right]}$
5	Policy-5 (OUT + net stock smoothing + on-order inventory smoothing + order quantity smoothing)	$\dfrac{z^{L+1}\left[\beta[z-(1-\alpha)]+\alpha(z-1)[1+\beta p\sqrt{1+k+L}+\gamma(k+L)+(1-\delta)]\right]}{z^2 z^L - z\left[z^L\left(\alpha(\delta-\gamma)-2\delta\right)-\alpha(\beta-\gamma)\right]-z^L\left[2\delta-\alpha\delta+1\right]}$

2.2. Preparation of frequency response plots

The demand pattern (input) to the system can be considered as a combination of (N/2-1) sine waves with each sine wave having a different frequency, amplitude (A) and phase angle. The amplitude of replenishment orders (output) which is also in sinusoidal form with matching frequency, but the difference in amplitude and phase angle, get de-amplified or amplified depends on the nature of transfer function derived for the inventory policy followed in the supply chain system. Then the amplitude ratio (U) can be defined as the ratio of the amplitude of replenishment orders to the amplitude of the customer demand. Frequency response graphs are generated by plotting amplitude ratios for sine waves (which corresponds to the input demand pattern) against the frequencies ranging from 0 to π radians per sample interval. Frequency response plot helps to identify whether an inventory policy generates bullwhip effect or not. The magnitude of amplitude ratio greater than unity denotes that the inventory policy causes amplification of order variance in the supply chain.

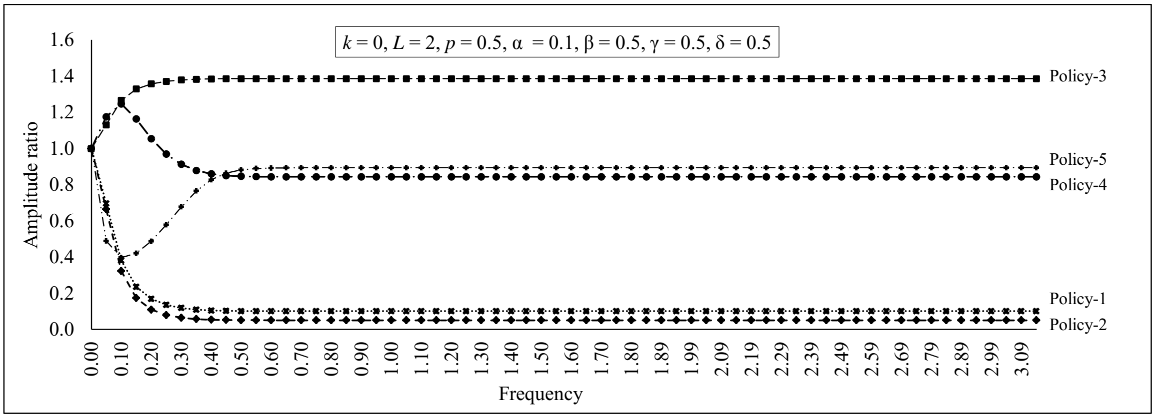
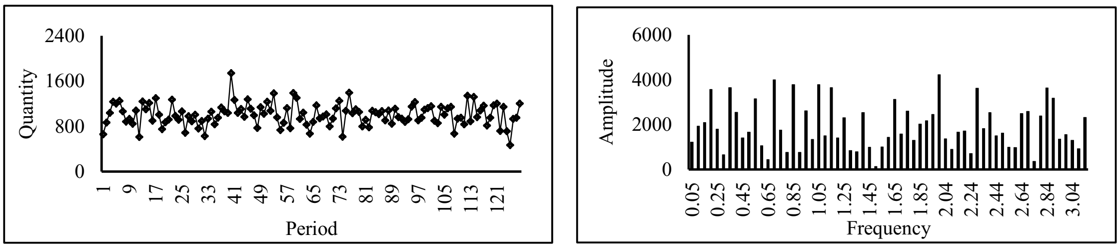


Fig.1. Frequency response plots for inventory policies

The preparation of the frequency response plot for a particular inventory policy involves two steps: (i) put $z = e^{i\omega t}$ in transfer function which is developed in the *z*-space, and (ii) calculation of magnitude of transfer function value in the complex plane [17]. The frequency response plots of inventory policies are given in Fig.1. As frequency increases, the amplitude ratio of Policy-1 and Policy-2 get decreases up to a particular value and then remain constant (the reduction is more for Policy-2, due to introduction of order smoothing parameter δ). The amplitude ratio is very less than unity (0.1 for Policy-1 and 0.05 for Policy-2), which means that the order variances get de-amplified and this policy causes anti-bullwhip/de-whip effect in supply chains. For Policy-3 (OUT policy), the amplitude ratio get increases as frequency increases and remains constant after a certain frequency. The amplitude ratio is greater than unity for OUT policy. For Policy-4, which is a variant of OUT policy, the introduction of smoothing parameters (net stock smoothing parameter (β) and on-order inventory smoothing parameter (γ)) reduce the amplitude ratio up to unity and so the order variance amplification under this policy is less than that of OUT policy. Policy-5 is another variant of OUT policy, which is formulated by introducing the order smoothing parameter (δ) to the Policy-4. The performance of Policy-5 is better than OUT but poorer than Policy-4.

2.3. Spectral analysis of demand patterns

For frequency response analysis, the input demand signal (time domain) is decomposed into some sinusoidal waves (frequency domain). Spectral analysis is the mathematical technique used for decomposing the time domain demand signal to sine waves, and this is achieved by Fast Fourier Transform method. The decomposed parts have a



(a) Demand-time diagram (b) Demand periodogram

Fig. 2. Demand-time diagram and demand periodogram for a constant demand pattern

constant term and (N/2-1) sine waves with a particular frequency, amplitude, and phase angle. In analogy with the demand-time diagram, the amplitude-frequency diagram (termed as 'periodogram') can be plotted by taking into account the amplitude associated with each sine wave against their corresponding frequency [6]. The demand-time diagram and corresponding demand periodogram under a constant demand pattern is shown in Fig. 2.

3. Bullwhip effect quantification in supply chain

3.1. Supply chain model

For this study, a single product four-stage serial supply chain is considered. The members in the stages are retailer (i = 1), wholesaler (i = 2), distributor (i = 3) and factory (i = 4). In this supply chain model, the end customer places an order to the retailer, and the retailer to the wholesaler and so on. Finally the factory places an order for production and manufactures the same. It is assumed that the factory has sufficient raw materials to manufacture the product. Every stage in a supply chain can have either customer or supplier relationship with the other. Each stage takes two decisions in each period, namely, the size of the shipment to be supplied to its customer and size of the order to be placed with its supplier.

3.2. Transfer function approach to supply chain

By applying the concepts of control system engineering, the transfer functions can be derived for any stage of a supply chain. For a decentralized multi-stage serial supply chain, the transfer function for the i^{th} stage can be i times the single stage transfer function [7] (the assumption is that the policy parameters and lead time are the same for all stages). The frequency response plots prepare for each stage of the supply chain using the derived transfer functions under all inventory policies. As an example, Fig. 3 presents the frequency response plots for each stage of the supply chain under OUT policy. In this figure the results are matched with the findings of Dejonckheere et al. [7] which also validate our study.

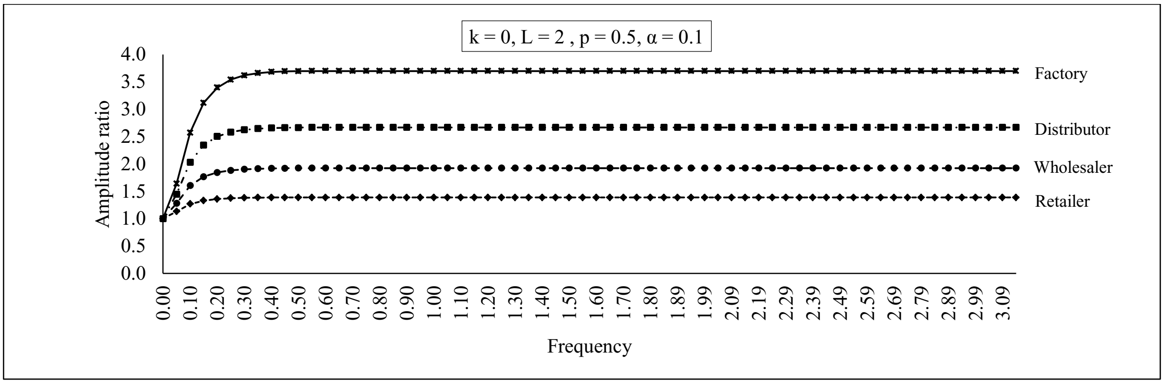


Fig. 3. Frequency response plots for each stage in the supply chain under OUT policy

3.3. Bullwhip effect measures

Order rate variance ratio is a common measure for bullwhip effect quantification and which is proposed by Chen et al. [3]. It is termed as the ratio of the variance of orders placed to the variance of customer demand. Dejonckheere et al. [6] showed that for an inventory policy, the amplitude ratios (U) obtained from a frequency response plot of the inventory policy and the amplitudes (A) obtained from the periodogram of the demand pattern under which the inventory policy is analyzed can be used to estimate the order rate variance ratio. The estimated order rate variance ratio can be denoted using the following equation:

$$OVR^i = \frac{Var^i}{Var^C} = \frac{\sum_{x=1}^{N/2-1} A_x^2 (U_x^2)_i}{\sum_{x=1}^{N/2-1} A_x^2} \tag{11}$$

For stages i = 1,..,4, order rate variance ratios are estimated which shows the amplification of order variance for each stage in a supply chain with respect to the customer demand. So it represents a stage-wise measure for bullwhip effect. Bullwhip slope is a network measure to quantify the bullwhip effect in supply chain and it indicates the nature of order variance amplification (slow or fast) within the supply chain [18]. The equation for calculating bullwhip slope is as follows:

$$BwSl = \frac{m^S \sum_{i=1}^{m^S} E^i OVR^i - \left(\sum_{i=1}^{m^S} E^i \sum_{i=1}^{m^S} OVR^i \right)}{m^S \sum_{i=1}^{m^S} (E^i)^2 - \left(\sum_{i=1}^{m^S} E_i \right)^2} \tag{12}$$

3.4. Analysis

The retailer faces a constant customer demand with mean 1000 and standard deviation 100 for the entire demand period (*N*). The total number of demand periods considered is 128. So the number of sine waves in the demand signal is 63 (*N*/2 -1). We have carried out the spectral analysis and demand-time diagrams, and amplitude-frequency diagrams are generated. The various other parameters selected for the analysis are as follows: k = 0, L= 2, p = 0.5 and α = 0.1. The performance of the supply chain in terms of order rate variance ratios and bullwhip slope is also estimated through spreadsheet simulation model. To validate our simulation results with the analytical study of Chen et al. [4] some modifications are made in the simulation model. The safety stock equated to 0 instead of $pF_t\sqrt{1+k+L}$ and order-up-to level estimated for $k+L$ periods instead of $k+L+T$ periods. Then the lead time is increased by 1 and which is termed as safety lead time [5,6,7]. Thus, for a production/ delivery lead time of 2 periods, the total lead time (L_T) will be 4 periods (2 periods delivery lead time + 1 period safety lead time + 1 period order delay due to a sequence of events). The order rate variance ratios estimated using both control theoretic and simulation methods are shown in Fig.4. The bullwhip slope estimated using two methods are given in Table 2.

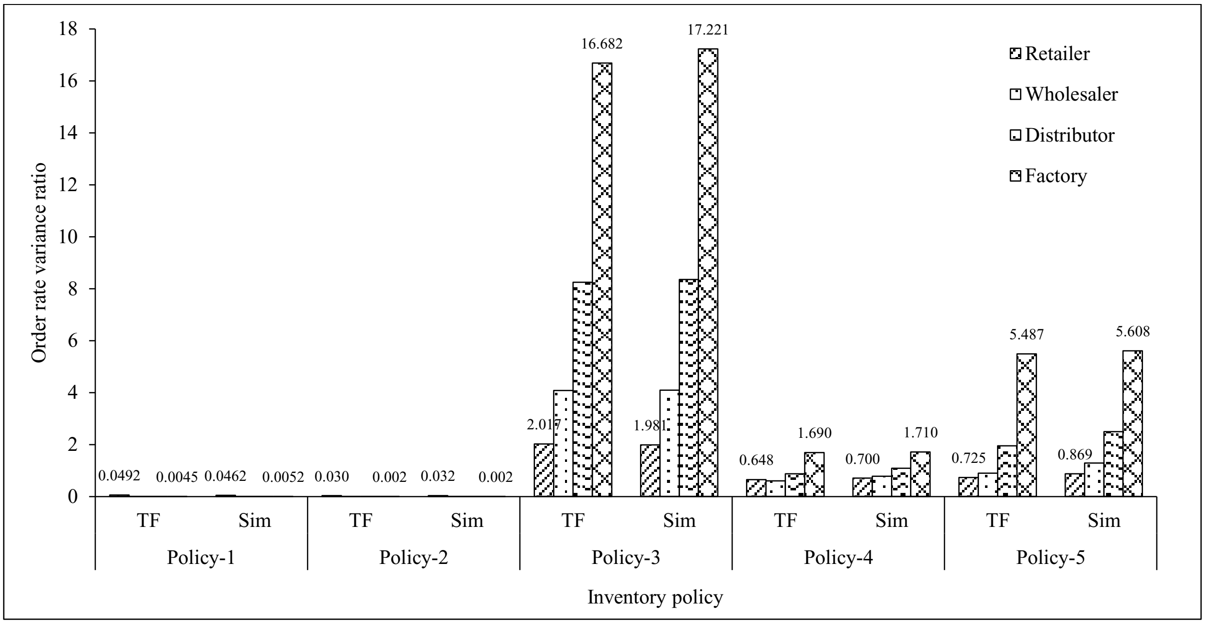


Fig. 4. Order rate variance ratio for various stages of supply chain under different policies using transfer function and simulation models

Table 2. Bullwhip slope of supply chain under different policies using transfer function and simulation models

Methodology	Policy-1	Policy-2	Policy-3	Policy-4	Policy-5
Transfer function method (TF)	-0.0145	-0.0088	4.8161	0.3400	1.5340
Simulation method (Sim)	-0.0131	-0.0094	4.9984	0.3352	1.5424

It is observed that under Policy-1 and Policy-2 (both are forecast based inventory policies) the order variance gets de-amplified from downstream stage to upstream stage. As the order quantity is determined based on the forecast, the smoothing of demand forecast results in a reduction of order variance. The bullwhip slope estimate found to be negative under these policies, which indicates the de-whip effect in the supply chain. The managerial implication is that de-whip effect results in more stable production planning and scheduling at the factory stage [19]. For the third policy (OUT), the order variance increases from downstream stage to upstream stage. So the OUT policy and its variants cause bullwhip generation in a serial supply chain; while the order variance amplification is more under the OUT policy than its variants.

4. Conclusions

In a supply chain, many factors cause the occurrence of bullwhip effect and its casualty on supply chain performance depend on the nature and degree of each factor. The present study focuses on the control system engineering approach to supply chain. Initially, we have derived the transfer function models of various inventory policies. Then the bullwhip effect performance measures such as order rate variance ratio and bullwhip slope are quantified in a serial supply chain under these policies using control theory approach. The effectiveness of the approach is evaluated in comparison with the simulation results. A possible extension of this study is the quantification of bullwhip effect in the divergent type supply chains using control theoretic approach.

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